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LETTER TO THE EDITOR

Non-existence of off-critical Yang–Baxter solutions for exceptional Lie algebra models

Ronald Rietman

Instituut voor Theoretische Fysica, Universiteit van Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands

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Abstract. It is shown that Pasquier's solution of the Yang–Baxter equations for the models based on the exceptional Lie algebras E_6, E_7, E_8 and their affine counterparts does not admit an off-critical extension in the space of weights of the configurations that satisfy the adjacency conditions imposed by the Dynkin diagram.

In the past decade considerable progress has been made in finding solutions of the Yang–Baxter equations (YBE). An important class of models for which Boltzmann weights satisfying the YBE have been found are the interaction-around-a-face (IRF) models on a square lattice. These models are defined as follows: take a finite square lattice $\mathscr{L}$ and a set $\boldsymbol{h}$ of 'heights'. A *configuration* is defined as a map $\mathscr{L} \to \boldsymbol{h}$. This model is of the IRF type when the Boltzmann weight of a configuration is given by the product over all faces of $\mathscr{L}$ of a face weight function W that depends only on the heights of the four sites around a face. We will take this face weight function the same for all faces of $\mathscr{L}$.

The models we will consider are restricted in two ways: the number of elements of $\boldsymbol{h}$ is finite and there is a function $C:\boldsymbol{h}\times\boldsymbol{h}\to\{0,1\}$ that indicates whether two neighbouring sites i and j of $\mathscr{L}$ are allowed to have the heights h_i and h_j: if $C(h_i, h_j)=1$ they are, if $C(h_i, h_j)=0$ they are not and the Boltzmann weights of corresponding configurations around a face equal zero. The function C can be represented by a symmetric $L\times L$ matrix, where L is the number of elements of $\boldsymbol{h}$, or by a graph with L vertices, each vertex corresponding to a single height. When $C(h', h)=1$ we draw a bond between the vertex representing height h and the one representing height h'. The number of allowed face configurations is given by $c_4=\operatorname{tr} C^4$.

For IRF models the definition of a transfer matrix is obvious [1]. A sufficient condition for transfer matrices to commute are the Yang–Baxter equations (YBE):

$$\forall a,\dots,f\in\boldsymbol{h}: \sum_{g\in\boldsymbol{h}} W_1\begin{pmatrix} a & g\\ b & c\end{pmatrix} W_2\begin{pmatrix} f & e\\ a & g\end{pmatrix} W_3\begin{pmatrix} e & d\\ g & c\end{pmatrix}$$

$$=\sum_{g\in\boldsymbol{h}} W_3\begin{pmatrix} f & g\\ a & b\end{pmatrix} W_2\begin{pmatrix} g & d\\ b & c\end{pmatrix} W_1\begin{pmatrix} f & e\\ g & d\end{pmatrix}. \tag{1}$$

There are $c_6\equiv\operatorname{tr} C^6$ YBE, one for each allowe set $\{a,\dots,f\}$.

A solution of the YBE can be found for arbitrary graphs [2, 3] and is given by

$$W_i\begin{pmatrix} d & c\\ a & b\end{pmatrix} = w_0\left(\begin{matrix} d & c\\ a & b\end{matrix}\middle| u_i\right) \tag{2}$$

where

$$w_0\begin{pmatrix} d & c \\ a & b \end{pmatrix}\!\left| u \right) = f(\lambda - u)\delta_{ac} + f(u)\left(\frac{s(a)s(c)}{s(b)s(d)}\right)^{1/2}\delta_{bd} \tag{3}$$

$$u_3 = u_2 - u_1 \tag{4}$$

and s is an eigenvector of C with non-negative eigenvalue Λ:

$$\forall h: \qquad \sum_{h' \in \boldsymbol{h}} C(h, h')s(h') = \Lambda s(h). \tag{5}$$

The eigenvalue Λ and crossing parameter λ are related through

$$\Lambda = \begin{cases} 2\cos\lambda & \text{if } 0 \leqslant \Lambda < 2 \\ 2\cosh\lambda & \text{if } \Lambda > 2 \end{cases} \tag{6}$$

and the function f is given by

$$f(x) = \begin{cases} \sin x/\sin\lambda & \text{if } 0 \leqslant \Lambda < 2 \\ \sinh x/\sinh\lambda & \text{if } \Lambda > 2. \end{cases} \tag{7}$$

Considering the limit $\Lambda \to 2$ in these expressions and rescaling the spectral parameter it is easily seen that for $\Lambda = 2$ we can take $f(x) = x$ and $\lambda = 1$ in equation (3).

Since all matrix elements of C are non-negative, its largest eigenvalue is positive and so are all elements of the corresponding eigenvector (Perron–Frobenius theorem). Taking s to be this eigenvector thus leads to real non-negative Boltzmann weights in the regime $0 \leqslant u \leqslant \lambda$.

The matrices C with largest eigenvalue $\Lambda_{\max} < 2$ can be classified [4]: the corresponding graphs are the Dynkin diagrams of the Lie algebras of types A, D and E, so the corresponding $\boldsymbol{h}$ is the set of simple roots. The corresponding models were considered by Pasquier [5] in the context of the ADE-classification of modular invariant partition functions.

The largest eigenvalue $\Lambda_{\max}$ of these matrices can be parametrized by an integer m that depends on the Lie algebra

$$\Lambda_{\max} = 2\cos\left(\frac{\pi}{m+1}\right). \tag{8}$$

Pasquier's solution describes *critical* models, but for the models of the A and D series it is possible to extend Pasquier's solution off the critical manifold. These restricted-solid-on-solid (RSOS) models [6] have Boltzmann weights that are parametrized in terms of theta functions of which the elliptic nome is zero at criticality, so that Pasquier's circular parametrization is recovered. For the exceptional models no off-critical extensions are known, although Pasquier, in his first *J. Phys. A* letter [5], claimed 'Clearly the E series can be handled similarly'. In this letter we address the question whether this really is true.

We have investigated this systematically by perturbing around the critical solution

$$W_i\begin{pmatrix} d & c \\ a & b \end{pmatrix} = w_0\begin{pmatrix} d & c \\ a & b \end{pmatrix}\!\left| u_i \right) + \delta W_i\begin{pmatrix} d & c \\ a & b \end{pmatrix} \tag{9}$$

and linearizing the YBE (1) in the δW_i. These linearized YBE can be rewritten in matrix notation as

$$\mathbf{M} \cdot \delta \boldsymbol{W} = 0 \tag{10}$$

where the matrix **M** is c_6 by $3c_4$ and depends on the spectral parameters. We must look for the nullspace of **M**, $M_0 \equiv \ker \mathbf{M}$.

In this nullspace there are vectors corresponding to transformations within the critical manifold. These are easily determined and fall into three classes.

(i) Multiplication of the weights by constant factors that are independent of the configuration:

$$W_i\begin{pmatrix} d & c \\ a & b \end{pmatrix} \to \rho_i W_i\begin{pmatrix} d & c \\ a & b \end{pmatrix}. \tag{11}$$

This gives three independent parameters.

(ii) Transformations of the spectral parameters:

$$u_i \to v_i \qquad \text{with} \qquad v_3 = v_2 - v_1. \tag{12}$$

This gives two independent parameters.

(iii) *Gauge transformations*, i.e. transformations on the face weights that leave the Boltzmann weight of an entire configuration invariant, so that also the partition function is invariant. A general gauge transformation can be written as

$$W_i\begin{pmatrix} d & c \\ a & b \end{pmatrix} \to \frac{F_i(a,b)}{F_i(d,c)}\frac{G_i(d,a)}{G_i(c,b)} W_i\begin{pmatrix} d & c \\ a & b \end{pmatrix} \tag{13}$$

where F_i and G_i are arbitrary functions $\boldsymbol{h} \times \boldsymbol{h} \to C\backslash\{0, \infty\}$. This gauge transformation leaves the YBE invariant provided we take

$$F_2 = F_1 \qquad F_3 = G_1 \qquad G_3 = G_2 \tag{14}$$

so this gives $3c_2$ parameters, where $c_2 \equiv \operatorname{tr} C^2$ is the number of allowed nearest-neighbour pairs.

The infinitesimal versions of the above transformations span a linear space $M_{0\text{crit}}$ of 'critical integrable perturbations' that is a subspace of M_0:

$$M_{0\text{crit}} \subseteq M_0. \tag{15}$$

The dimension of $M_{0\text{crit}}$ is $\leqslant 3c_2 + 5$.

If $M_0 = M_{0\text{crit}}$, it is obviously impossible to find an off-critical extension of the critical solution since then any integrable perturbation is critical. There are non-critical integrable perturbations only if M_0 is strictly larger than $M_{0\text{crit}}$.

We used the 'singular value decomposition' routine from [7] to calculate M_0, its dimension and the dimension of $M_{0\text{crit}}$ for a few values of the spectral prameters:

Lie algebra	m	c_2	c_4	c_6	dim $M_{0\text{crit}}$	dim M_0
E_6	11	10	30	106	27	27
E_7	17	12	36	126	32	32
E_8	29	14	42	146	37	37
$E_6^{(1)}$	∞	12	36	132	32	32
$E_7^{(1)}$	∞	14	42	146	37	37
$E_8^{(1)}$	∞	16	48	166	42	42
A_4	5	6	14	36	17	18

This shows that $M_0 = M_{0\text{crit}}$ for the exeptional Lie algebra models and their affine cousins, implying that there is no off-critical extension. The result for A_4 shows that there is one off-critical extension of this model. This implies that the known extension is the only one.

Finally we should stress that we only investigated the existence of off-critical extensions in the space of weights of those configurations that satisfy the adjacency conditions imposed by the various Dynkin diagrams. The existence of off-critical extensions with new heights or looser adjacency conditions cannot be excluded.

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